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On the inverse eigenvalue problem for nonnegative matrices of order two to five

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ABSTRACT

In this paper, for a given set of numbers with special conditions, we construct a nonnegative matrix such that the given set is its spectrum.

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers in order that it be the spectrum of a nonnegative matrix. In this case, one says that σ is realizable and a nonnegative matrix A with spectrum σ is said to realize σ and it is referred to as a realizing matrix. A number of necessary conditions for realizability are known, as well as a number of sufficient conditions. In many cases, sufficiency is established by direct construction of a realizing matrix [1–6].

In terms of n, complete solutions to the NIEP are available only for $n \leq 4$. The case $n = 2$ is trivial-necessary and sufficient conditions are that λ_1 and λ_2 are both real and their sum is nonnegative. The first non-trivial result on realizability was found by Suleimanova who showed that if, in the list σ, all the λ_i are real and all but one are negative, then σ is realizable if and only if the sum of its elements is nonnegative [7]. Proofs of her result, including the fact that a realizing matrix can be chosen to be

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a companion matrix, can be found in Friedland [9]. The problem has been solved for $n = 3$ by Lowey and London [3]. The cases $n = 4$ and $n = 5$ have been solved for matrices with trace zero by Reams [4] and Laffey and Meehan [5], respectively. So, for real spectra, complete constructive solutions to the NIEP are available for $n \leq 4$.

For the case of non-real spectra σ for $n = 4$, complete solutions are available through work of Laffey and Meehan [5] (see Meehan's 1998 doctoral thesis (National University of Ireland, Dublin [12])) and, independently, that of Torre-Mayo et al. by analyzing coefficients of the characteristic polynomial. EBL digraphs [11]. Rojo and Soto found a necessary and sufficient condition for $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ to be the spectrum of some circulant nonnegative matrix [13]. In [10] Šmigoc started with a realizable list of real numbers and obtained a realizable list that contains elements that are not real.

Through this paper the following notation is used. The spectral radius of nonnegative matrix A denoted by $\rho(A)$. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. In addition s_k the kth power sum of the eigenvalues λ_i and in the list σ, λ_1 is the Perron element.

Some necessary conditions on the list of complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.

(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron–Frobenius theorem).
(2) The list σ is closed under complex conjugation.
(3) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$.
(4) $s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality) [3,8].

In Section 2 we present a theorem that is similar to Lemma 5 of [2] and by using this theorem, we construct a $n \times n$ nonnegative matrix for a given set which satisfies special conditions in a recursive method for $n \leq 5$.

Although we have a recursive method for solving the NIEP, it must be said that constructive methods presented do not cover all realizable spectra of n elements except when $n = 2$.

2. Construction

Theorem 2.1. *Let B be a $m \times m$ nonnegative matrix, $M_1 = \{\mu_1, \mu_2, \ldots, \mu_m\}$ be its eigenvalues and μ_1 be Perron eigenvalue of B. Also assume that A is a $n \times n$ nonnegative matrix in following form:*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix},$$

where A_1 is a $(n-1) \times (n-1)$ matrix, a and b are arbitrary vectors in $\mathbb{R}^{n-1}$ and $M_2 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the set of eigenvalues of A. Then there exist a $(m+n-1) \times (m+n-1)$ nonnegative matrix such that $M = \{\mu_2, \ldots, \mu_m, \lambda_1, \lambda_2, \ldots, \lambda_m\}$ is its eigenvalues.

Proof. Let s be the normalized eigenvector of B corresponding to Perron eigenvalue μ_1. Now, we find an $m \times (m-1)$ matrix V_1, such that $Y_1 = \begin{pmatrix} s & V_1 \end{pmatrix}$ be an unitary matrix. Therefore

$$BY_1 = \begin{pmatrix} \mu_1 s & BV_1 \end{pmatrix},$$

and then we have

$$B_1 = Y_1^* B Y_1 = \begin{pmatrix} \mu_1 s s^* & s^* B V_1 \\ \mu_1 V_1^* s & V_1^* B V_1 \end{pmatrix} = \begin{pmatrix} \mu_1 & \star & \ldots & \star \\ 0 & & & \\ \vdots & & \hat{B} & \\ 0 & & & \end{pmatrix},$$

where $\hat{B} = V_1^* B V_1$ has order $(m-1) \times (m-1)$. By the above relation, we obtain that the elements of $\{\mu_2, \ldots, \mu_m\}$ are eigenvalues of $\hat{B}$. Furthermore, by Schur decomposition theorem, there exist unitary matrix V_2 of order $m-1$, such that $V_2^* \hat{B} V_2 = \hat{T}_B$, where $\hat{T}_B$ is upper triangular matrix and the elements of its main diagonal are eigenvalues $\hat{B}$. Now we define

$$Y_2 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & V_2 & & \\ 0 & & & \end{pmatrix}.$$

Since V_2 is unitary, then Y_2 is an unitary matrix, we have

$$Y_2^* B_1 Y_2 = Y_2^* (Y_1^* B Y_1) Y_2 = (Y_1 Y_2)^* B (Y_1 Y_2) = Y^* B Y \quad (\text{if } Y = Y_1 Y_2)$$

$$Y = Y_1 Y_2 = \begin{pmatrix} s & V_1 V_2 \end{pmatrix} = \begin{pmatrix} s & T \end{pmatrix}, \quad Y^* = \begin{pmatrix} s^* \\ T^* \end{pmatrix} \quad (\text{if } V_1 V_2 = T)$$

Y is a $m \times m$ unitary matrix and the order of matrix T is $m \times (m-1)$. From the above relations we have:

$$\begin{aligned} YY^* &= ss^* + TT^* = I_m, \\ Y^* Y &= \begin{pmatrix} s^* s & s^* T \\ T^* s & T^* T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{m-1} \end{pmatrix}. \end{aligned} \tag{2.1}$$

$$Y^* B Y = \begin{pmatrix} s^* B s & s^* B T \\ T^* B s & T^* B T \end{pmatrix} = \begin{pmatrix} \mu_1 & \bigstar \\ 0 & \hat{T}_B \end{pmatrix} = T_B. \tag{2.2}$$

It is clear that T_B is an upper triangular matrix and the elements of its main diagonal are elements of the set M_1.

By the Schur decomposition theorem, there exists an unitary matrix X such that $X^* A X = T_A$, where T_A is an upper triangular matrix with elements of set M_2 on its main diagonal. The matrices X and X^* can be partitioned as follows:

$$X = \begin{pmatrix} V \\ K \end{pmatrix} \quad \text{and} \quad X^* = \begin{pmatrix} V^* & K^* \end{pmatrix},$$

where V and K are of order $(n-1) \times n$ and $1 \times n$, respectively. Since X is an unitary matrix, then it is easy to verify that;

$$\begin{aligned} XX^* &= \begin{pmatrix} VV^* & VK^* \\ KV^* & KK^* \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix}, \\ X^* X &= V^* V + K^* K = I_n. \end{aligned} \tag{2.3}$$

By relations (2.3) and $X^* A X = T_A$, we have,

$$T_A = X^* A X = V^* A_1 V + K^* b^T V + V^* a K + K^* \mu_1 K. \tag{2.4}$$

We consider two matrices Z and Z^* and a nonnegative matrix C of order $(m+n-1) \times (m+n-1)$ in the following form:

$$Z = \begin{pmatrix} V & 0 \\ sK & T \end{pmatrix}, \quad Z^* = \begin{pmatrix} V^* & K^*s^* \\ 0 & T^* \end{pmatrix}, \quad C = \begin{pmatrix} A_1 & as^* \\ sb^T & B \end{pmatrix}.$$

Using relations (2.1) and (2.3), it is easy to show that Z is an unitary matrix. Now by relations (2.1)–(2.4), we can calculate Z^*CZ as below:

$$Z^*CZ = \begin{pmatrix} V^*A_1V + K^*b^TV + V^*as^*sK + K^*s^*BsK & V^*as^*T + K^*s^*BT \\ T^*sb^TV + TBsK & T^*BT \end{pmatrix} = \begin{pmatrix} T_A & \star \\ 0 & \hat{T}_B \end{pmatrix} = T_C,$$

where T_C is an upper triangular matrix and the elements of its main diagonal are the elements of M. On the other hand by the relation above, C and T_C are similar, therefore C solves the problem which completes the proof. $\square$

Remark 2.2. If we take matrices A and B in the Theorem (2.1) to be symmetric, it is obvious that the matrix C constructed in the theorem is also symmetric. Furthermore let A and B be normal. Hence, matrices T_A and $\hat{T}_B$ in the proof of Theorem (2.1) are diagonal. It is easy to see that in this case T_C is also diagonal. We conclude that normal matrices A and B give us a normal matrix C.

3. The case $n = 2$

Theorem 3.1. *Let $\sigma = \{\lambda_1, \lambda_2\}$ be a set of two real numbers such that $\lambda_1 \geqslant |\lambda_2|$. Then σ is the set of eigenvalues of a nonnegative matrix.*

Proof. σ has only one of following cases:

(a) If $\lambda_2 \geqslant 0$, then $A = \mathrm{diag}(\lambda_1, \lambda_2)$ is a solution of problem.

(b) If $\lambda_2 < 0$, then the matrix

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix} \tag{3.1}$$

solves the problem. $\square$

4. The case $n = 3$

Theorem 4.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real and complex numbers and*

$$\begin{aligned} p &= \lambda_1\lambda_2\lambda_3, \\ \alpha_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ \alpha_2 &= \lambda_1 - \lambda_2 - \lambda_3 - |\lambda_2|^2 - 1, \\ \alpha_3 &= \lambda_1 - |\lambda_2|^2, \\ \alpha_4 &= \lambda_2 + \lambda_3 + |\lambda_2|^2. \end{aligned}$$

If σ satisfies in the following conditions:

$$\lambda_1 + \lambda_2 + \lambda_3 \geqslant 0, \tag{4.1}$$

$$\sigma = \bar{\sigma}, \tag{4.2}$$

$$\lambda_1 \in \mathbb{R}, \quad \lambda_1 \geqslant |\lambda_i| \quad i = 2, 3, \tag{4.3}$$

$$\textit{if } (\lambda_1, \lambda_2 \notin \mathbb{R}, \alpha_1 > 0) \longrightarrow \alpha_2, \alpha_3, \alpha_4 \geqslant 0. \tag{4.4}$$

Then there exists the nonnegative matrix C, such that σ is its spectrum.

Proof. At first, we assume that all elements of σ are real numbers. By the above conditions we have the following cases:

(a) If $\lambda_2, \lambda_3 \geqslant 0$, then the nonnegative matrix $C = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a solution of our problem.

(b) If $\lambda_2, \lambda_3 < 0$, since two nonnegative matrices A and B with spectrum $\sigma_1 = \{\lambda_1, \lambda_2\}$, $\sigma_2 = \{\lambda_1 + \lambda_2, \lambda_3\}$ in the following form:

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix}$$

respectively, satisfy in the conditions of Theorem 2.1, and $s = \left(\frac{-\lambda_3}{\sqrt{1+\lambda_3^2}}, \frac{1}{\sqrt{1+\lambda_3^2}}\right)^T$ is the normalized eigenvector associated to Perron eigenvalue, $\lambda = \lambda_1 + \lambda_2$, of nonnegative matrix B, then by this theorem the nonnegative matrix

$$C = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\sqrt{1+\lambda_3^2}} & \frac{-\lambda_1\lambda_2}{\sqrt{1+\lambda_3^2}} \\ \frac{-\lambda_3}{\sqrt{1+\lambda_3^2}} & 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ \frac{1}{\sqrt{1+\lambda_3^2}} & 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix} \tag{4.5}$$

is a solution of the problem.

(c) If $\lambda_2 < 0$ and $\lambda_3 \geqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_3 \end{pmatrix} \tag{4.6}$$

is a solution of our problem, where A is matrix (3.1) and a, b are zero vectors with dimension of 2×1.

Now let λ_2, λ_3 be a conjugate complex pair. One of the following cases will happen.

(d) If $\alpha_1 \leqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix}, \tag{4.7}$$

solves the problem.

(e) If $\alpha_1 > 0$, then by condition (4.4), we must have $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$. So the nonnegative matrix

$$C = \begin{pmatrix} \alpha_3 & \alpha_2|\lambda_2|^2 & 1 \\ 1 & \alpha_4 & 0 \\ 0 & p & 0 \end{pmatrix} \tag{4.8}$$

is a solution of our problem.

(f) If $\alpha_1 > 0$ and $\alpha_2 < 0$, then by [3] we can find the nonnegative matrix C. $\square$

5. The case $n = 4$

Theorem 5.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of real and complex numbers and*

$$\begin{aligned} p &= \lambda_1\lambda_2\lambda_3, \\ \alpha_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ \alpha_2 &= \lambda_1 - \lambda_2 - \lambda_3 - |\lambda_2|^2 - 1, \\ \alpha_3 &= \lambda_1 - |\lambda_2|^2, \\ \alpha_4 &= \lambda_2 + \lambda_3 + |\lambda_2|^2. \end{aligned}$$

If σ satisfies the following conditions, then there exist a nonnegative matrix C, such that σ is its spectrum.

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant 0, \tag{5.1}$$

$$\sigma = \bar{\sigma}, \tag{5.2}$$

$$\lambda_1 \in \mathbb{R}, \quad \lambda_1 \geqslant |\lambda_i|; \quad i = 2, 3, 4 \tag{5.3}$$

and if λ_2, λ_3 be a conjugate complex pair and λ_4 be a real number, then in addition to conditions (5.1)–(5.3), σ must satisfy the following conditions:

$$\textit{if } (\alpha_1 \leqslant 0, \lambda_4 > 0) \longrightarrow \lambda_1 + \lambda_2 + \lambda_3 \geqslant 0, \tag{5.4}$$

$$\textit{if } (\alpha_1 > 0, \lambda_4 \geqslant 0) \longrightarrow \alpha_2, \alpha_3, \alpha_4 \geqslant 0, \tag{5.5}$$

$$\begin{aligned} &\textit{if } (\alpha_1 > 0, \lambda_4 < 0) \longrightarrow \\ &\alpha_2 \geqslant 0 \textit{ and } \{(\alpha_3 \geqslant |\lambda_4|, \alpha_4 \geqslant 0) \textit{ or } (\alpha_3 \geqslant 0, \alpha_4 \geqslant |\lambda_4|)\}. \end{aligned} \tag{5.6}$$

Proof. At first, we assume that all elements of σ are real numbers. In accordance with the above conditions we consider the following cases:

(a) If $\lambda_2, \lambda_3, \lambda_4 \geqslant 0$, then the nonnegative matrix $C = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a desired matrix.

(b) If $\lambda_2, \lambda_3, \lambda_4 < 0$, then by using Theorem 2.1, we can construct the desired matrix, since A equal to the matrix in (4.5) with the spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $B = \begin{pmatrix} 0 & -(\lambda_1 + \lambda_2 + \lambda_3)\lambda_4 \\ 1 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix}$ satisfy the conditions of that theorem. The normalized vector associated to Perron eigenvalue of $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ of matrix B has form $s = \left(\frac{-\lambda_4}{\sqrt{1+\lambda_4^2}}, \frac{1}{\sqrt{1+\lambda_4^2}}\right)^T$. Consequently, the following nonnegative matrix is a solution of our problem

$$C = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\sqrt{1+\lambda_3^2}} & \frac{\lambda_1\lambda_2\lambda_4}{\sqrt{1+\lambda_3^2}\sqrt{1+\lambda_4^2}} & \frac{-\lambda_1\lambda_2}{\sqrt{1+\lambda_3^2}\sqrt{1+\lambda_4^2}} \\ \frac{-\lambda_3}{\sqrt{1+\lambda_4^2}} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\sqrt{1+\lambda_4^2}} \\ \frac{-\lambda_4}{\sqrt{1+\lambda_3^2}\sqrt{1+\lambda_4^2}} & \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & 0 & -(\lambda_1 + \lambda_2 + \lambda_3)\lambda_4 \\ \frac{1}{\sqrt{1+\lambda_3^2}\sqrt{1+\lambda_4^2}} & \frac{1}{\sqrt{1+\lambda_4^2}} & 1 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix}. \tag{5.7}$$

(c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4 \geqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_4 \end{pmatrix} \tag{5.8}$$

is a solution of this problem,where A is matrix (4.6) and a, b are zero vectors with dimension of 3×1.

(d) If $\lambda_2, \lambda_3 \leqslant 0$, $\lambda_4 > 0$ and at least for one of the eigenvalues λ_2 and λ_3, for example λ_3, we have $\lambda_3 + \lambda_4 \geqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 \\ 1 & \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3\lambda_4 \\ 0 & 0 & 1 & \lambda_3 + \lambda_4 \end{pmatrix}$$

is a solution of our problem.

(e) If $\lambda_2, \lambda_3 \leqslant 0$, $\lambda_4 > 0$ and we have $\lambda_2 + \lambda_4 \leqslant 0$, $\lambda_3 + \lambda_4 \leqslant 0$, then by [3] we consider $C = (c_{ij})$ in the following form:

$$C = U \begin{pmatrix} \lambda_2 & & & \\ & \lambda_3 & & \\ & & \lambda_4 & \\ & & & \lambda_1 \end{pmatrix} U^T,$$

where U is orthogonal matrix

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

We have also the following relations:

$$\begin{aligned}
c_{11} = c_{22} = c_{33} = c_{44} &= \tfrac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \geqslant 0, \\
c_{12} = c_{21} = c_{34} = c_{43} &= \tfrac{1}{4}(\lambda_1 - (\lambda_2 + \lambda_3) + \lambda_4) \geqslant 0, \\
c_{13} = c_{31} = c_{24} = c_{42} &= \tfrac{1}{4}((\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4)) \geqslant 0, \\
c_{14} = c_{41} = c_{23} = c_{32} &= \tfrac{1}{4}((\lambda_1 + \lambda_3) - (\lambda_2 + \lambda_4)) \geqslant 0.
\end{aligned}$$

So C is a nonnegative matrix and solve our problem.

Now let λ_2, λ_3 be the complex conjugate pair. One of the following cases will happen.

(f) If $\alpha_1 \leqslant 0$, $\lambda_4 > 0$, then by (5.4) we have $\lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$. Consequently the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_4 \end{pmatrix} \tag{5.9}$$

is a solution of our problem. where A is the matrix (4.7) and a, b are zero vector of dimension 3×1.

(g) If $\alpha_1 \leqslant 0$, $\lambda_4 \leqslant 0$, then by Theorem 2.1, we construct the solution of our problem. Whereas the nonnegative matrix (4.7) with the spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and the nonnegative matrix

$$B = \begin{pmatrix} 0 & -(\lambda_1+\lambda_2+\lambda_3)\lambda_4 \\ 1 & \lambda_1+\lambda_2+\lambda_3+\lambda_4 \end{pmatrix}$$

with the spectrum $\sigma_2 = \{\lambda_1+\lambda_2+\lambda_3, \lambda_4\}$ have the condition of matrices A and B of Theorem 2.1, respectively, and since the normalized vector associated to the Perron eigenvalue of $\lambda = \lambda_1+\lambda_2+\lambda_3$ of matrix B is $s = \left(\frac{-\lambda_4}{\sqrt{1+\lambda_4^2}}, \frac{1}{\sqrt{1+\lambda_4^2}}\right)^T$, then by this theorem, the following matrix is a solution of our problem

$$C = \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{1}{\sqrt{1+\lambda_4^2}} \\ \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{\alpha_1\lambda_4}{\sqrt{1+\lambda_4^2}} & 0 & -(\lambda_1+\lambda_2+\lambda_3)\lambda_4 \\ \frac{1}{\sqrt{1+\lambda_4^2}} & \frac{-\alpha_1}{\sqrt{1+\lambda_4^2}} & 1 & \lambda_1+\lambda_2+\lambda_3+\lambda_4 \end{pmatrix}. \tag{5.10}$$

(h) If $\alpha_1 > 0$, $\lambda_4 \geqslant 0$, then by (5.5) we must have $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$. Consequently the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_4 \end{pmatrix} \tag{5.11}$$

is a solution of our problem, where A is the matrix (4.8) and a, b are zero vector with dimension 3×1.

(i) If $\alpha_1 > 0$, $\lambda_4 < 0$, then by relation (5.6) we must have $\alpha_2 \geqslant 0$ and $\{(\alpha_3 \geqslant |\lambda_4|, \alpha_4 \geqslant 0)$ *or* $(\alpha_3 \geqslant 0, \alpha_4 \geqslant |\lambda_4|)\}$. If $\alpha_2, \alpha_4 \geqslant 0$ and $\alpha_3 \geqslant |\lambda_4|$, to construct the matrix solving our problem, first, recall that in the case $n = 3$, we constructed nonnegative matrices A and B with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_3, \lambda_4\}$, respectively, in the following form:

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & \alpha_4 & 1 \\ 1 & \alpha_2|\lambda_2|^2 & \alpha_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\alpha_3\lambda_4 \\ 1 & \alpha_3+\lambda_4 \end{pmatrix}$$

since the nonnegative matrices above satisfy conditions of Theorem 2.1, and the normalized eigenvector of the Perron eigenvalue α_3 of nonnegative matrix B is $s = \left(\frac{-\lambda_4}{\sqrt{1+\lambda_4^2}}, \frac{1}{\sqrt{1+\lambda_4^2}}\right)^T$, then by this theorem, the nonnegative matrix

$$C = \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & \alpha_4 & \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{1}{\sqrt{1+\lambda_4^2}} \\ \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{-\alpha_2\lambda_4|\lambda_2|^2}{\sqrt{1+\lambda_4^2}} & 0 & -\alpha_3\lambda_4 \\ \frac{1}{\sqrt{1+\lambda_4^2}} & \frac{\alpha_2|\lambda_2|^2}{\sqrt{1+\lambda_4^2}} & 1 & \alpha_3+\lambda_4 \end{pmatrix} \tag{5.12}$$

is a solution of our problem.

If $\alpha_2, \alpha_3 \geqslant 0$ and $\alpha_4 \geqslant |\lambda_4|$, then apply the preceding case by using matrices A and B with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_4, \lambda_4\}$, respectively, in the following form:

$$A = \begin{pmatrix} \alpha_3 & 1 & \alpha_2|\lambda_2|^2 \\ 0 & 0 & p \\ 1 & 0 & \alpha_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\alpha_4\lambda_4 \\ 1 & \alpha_4 + \lambda_4 \end{pmatrix},$$

whereas the normalized eigenvector associated to Perron eigenvalue α_4 of nonnegative matrix B has form $s = \left(\frac{-\lambda_4}{\sqrt{1+\lambda_4^2}}, \frac{1}{\sqrt{1+\lambda_4^2}}\right)^T$, then by Theorem 2.1, the nonnegative matrix

$$C = \begin{pmatrix} \alpha_3 & 1 & \frac{-\lambda_4\alpha_2|\lambda_2|^2}{\sqrt{1+\lambda_4^2}} & \frac{\alpha_2|\lambda_2|^2}{\sqrt{1+\lambda_4^2}} \\ 0 & 0 & \frac{-p\lambda_4}{\sqrt{1+\lambda_4^2}} & \frac{p}{\sqrt{1+\lambda_4^2}} \\ \frac{-\lambda_4}{\sqrt{1+\lambda_4^2}} & 0 & 0 & -\alpha_4\lambda_4 \\ \frac{1}{\sqrt{1+\lambda_4^2}} & 0 & 1 & \alpha_4 + \lambda_4 \end{pmatrix} \tag{5.13}$$

is a solution of our problem. □

6. The case $n = 5$

Theorem 6.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a set of real and complex numbers and*

$$\begin{aligned}
p &= \lambda_1\lambda_2\lambda_3, \\
\alpha_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\
\alpha_2 &= \lambda_1 - \lambda_2 - \lambda_3 - |\lambda_2|^2 - 1, \\
\alpha_3 &= \lambda_1 - |\lambda_2|^2, \\
\alpha_4 &= \lambda_2 + \lambda_3 + |\lambda_2|^2 \\
s_1' &= \lambda_1 + \lambda_2 + \lambda_3, \\
\alpha_5 &= s_1'(\lambda_4 + \lambda_5) + |\lambda_4|^2, \\
\alpha_6 &= s_1' - \lambda_4 - \lambda_5 - |\lambda_4|^2 - 1, \\
\alpha_7 &= s_1' - |\lambda_4|^2, \\
\alpha_8 &= \alpha_3(\lambda_4 + \lambda_5) + |\lambda_4|^2, \\
\alpha_9 &= \alpha_3 - \lambda_4 - \lambda_5 - |\lambda_4|^2 - 1, \\
\alpha_{10} &= \alpha_3 - |\lambda_4|^2, \\
\alpha_{11} &= \alpha_4(\lambda_4 + \lambda_5) + |\lambda_4|^2 \\
\alpha_{12} &= \alpha_4 - \lambda_4 - \lambda_5 - |\lambda_4|^2 - 1, \\
\alpha_{13} &= \alpha_4 - |\lambda_4|^2, \\
\alpha_{14} &= \lambda_4 + \lambda_5 + |\lambda_4|^2,
\end{aligned}$$

If σ satisfies the following conditions, then there exists the nonnegative matrix C, such that σ is its spectrum,

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0, \tag{6.1}$$

$$\sigma = \bar{\sigma}, \tag{6.2}$$

$$\lambda_1 \in \mathbb{R}, \quad \lambda_1 \geqslant |\lambda_i|; \quad i = 2, 3, 4, 5, \tag{6.3}$$

if all elements of σ are real, then we assume the following conditions in addition to the conditions (6.1)–(6.3),

$$\text{if } (\lambda_2, \lambda_3 < 0, \lambda_4, \lambda_5 \geqslant 0) \longrightarrow s'_1 \geqslant 0 \tag{6.4}$$

$$\text{if } (\lambda_2, \lambda_3, \lambda_4 < 0, \lambda_5 \geqslant 0) \longrightarrow s'_1 + \lambda_4 \geqslant 0 \tag{6.5}$$

If σ is not real and λ_2 and λ_3 are complex conjugates, while λ_4 and λ_5 are real, then we assume the following conditions in addition to (6.1)–(6.3).

$$\text{if } (\lambda_4, \lambda_5 \geqslant 0, \alpha_1 \leqslant 0) \longrightarrow s'_1 \geqslant 0, \tag{6.6}$$

$$\text{if } (\lambda_4, \lambda_5 \geqslant 0) \longrightarrow \alpha_2, \alpha_3, \alpha_4 \geqslant 0, \tag{6.7}$$

$$\text{if } (\lambda_4 < 0, \lambda_5 \geqslant 0, \alpha_1 \leqslant 0) \longrightarrow s'_1 + \lambda_4 \geqslant 0, \tag{6.8}$$

$$\text{if } (\lambda_4 < 0, \lambda_5 \geqslant 0, \alpha_2, \alpha_3, \alpha_4 \geqslant 0) \longrightarrow \alpha_3 + \lambda_4 \geqslant 0, \tag{6.9}$$

$$\begin{aligned} &\text{if } (\lambda_4 < 0, \lambda_5 \geqslant 0, \alpha_1 > 0) \longrightarrow \alpha_2 \geqslant 0, \{(\alpha_3 \geqslant |\lambda_4|, \alpha_4 \geqslant 0) \text{ or} \\ &\quad (\alpha_3 \geqslant 0, \alpha_4 \geqslant |\lambda_4|)\} \end{aligned} \tag{6.10}$$

$$\text{if } (\lambda_4, \lambda_5 < 0, \alpha_1 < 0) \longrightarrow s'_1 + \lambda_4 + \lambda_5 \geqslant 0, \tag{6.11}$$

$$\begin{aligned} &\text{if } (\lambda_4, \lambda_5 < 0, \alpha_2, \alpha_4 \geqslant 0) \longrightarrow \alpha_3 + \lambda_4 \geqslant |\lambda_5|, \\ &\text{if } (\lambda_4, \lambda_5 < 0, \alpha_2, \alpha_3 \geqslant 0) \longrightarrow \alpha_4 + \lambda_4 \geqslant |\lambda_5|, \end{aligned} \tag{6.12}$$

If λ_2, λ_3 and λ_4, λ_5 be two complex conjugate pairs, then the set of σ in addition to conditions (6.1)–(6.3) must satisfy the following conditions:

$$\textit{if } (\alpha_1, \alpha_5 \leqslant 0) \longrightarrow s'_1 \geqslant 0, \tag{6.13}$$

$$\textit{if } (\alpha_1 \leqslant 0, \alpha_5 > 0) \longrightarrow \alpha_6, \alpha_7, \alpha_{14} \geqslant 0, \tag{6.14}$$

$$\textit{if } (\alpha_1 > 0, \alpha_8 \leqslant 0) \longrightarrow \alpha_2, \alpha_3, \alpha_4 \geqslant 0, \alpha_3 + \lambda_4 + \lambda_5 \geqslant 0 \tag{6.15}$$

$$\textit{if } (\alpha_1 \leqslant 0, \alpha_8 > 0) \longrightarrow \alpha_2, \alpha_4, \alpha_9, \alpha_{10}, \alpha_{14} \geqslant 0 \tag{6.16}$$

$$\textit{if } (\alpha_1 > 0, \alpha_{11} \leqslant 0) \longrightarrow \alpha_2, \alpha_3, \alpha_4 \geqslant 0, \alpha_4 + \lambda_4 + \lambda_5 \geqslant 0, \tag{6.17}$$

$$\textit{if } (\alpha_1 \leqslant 0, \alpha_{11} > 0) \longrightarrow \alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{14} \geqslant 0, \tag{6.18}$$

Proof. At first, we assume that the all elements of σ are real numbers. Then by conditions above we consider the following cases:

(a) If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \geqslant 0$, then the nonnegative matrix $C = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is solution.

(b) Although this case is the case $n = 5$ of the original Suleimanova theorem, and, Friedland showed that one can take C to be a companion matrix in this case [6,9], we try to find another nonnegative matrix. If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 < 0$, then by Theorem 2.1, we construct solution. Whereas the nonnegative matrix (5.7) with the spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and nonnegative matrix

$$B = \begin{pmatrix} 0 & -(s'_1 + \lambda_4)\lambda_5 \\ 1 & s'_1 + \lambda_4 + \lambda_5 \end{pmatrix}$$

with eigenvalues $\sigma_2 = \{s'_1 + \lambda_4, \lambda_5\}$ have the conditions of matrices A and B of Theorem 2.1, respectively. The normalized eigenvector corresponding to Perron eigenvalue of $\lambda = s'_1 + \lambda_4$ of nonnegative matrix B has the form of $s = \left(\frac{-\lambda_5}{\sqrt{1+\lambda_5^2}}, \frac{1}{\sqrt{1+\lambda_5^2}}\right)^T$, then with setting $\mu = \sqrt{1+\lambda_3^2}$, $\beta = \sqrt{1+\lambda_4^2}$ and $\gamma = \sqrt{1+\lambda_5^2}$ by Theorem 2.1, the following nonnegative matrix is solution,

$$C = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\mu} & \frac{\lambda_1\lambda_2\lambda_4}{\mu\beta} & \frac{\lambda_1\lambda_2\lambda_5}{\mu\beta\gamma} & \frac{-\lambda_1\lambda_2}{\mu\beta\gamma} \\ \frac{-\lambda_3}{\mu} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\beta} & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_5}{\beta\gamma} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\beta\gamma} \\ \frac{-\lambda_4}{\mu\beta} & \frac{-\lambda_4}{\beta} & 0 & \frac{s_1'\lambda_4\lambda_5}{\gamma} & \frac{-s_1'\lambda_4}{\gamma} \\ \frac{-\lambda_5}{\mu\beta\gamma} & \frac{-\lambda_5}{\beta\gamma} & \frac{-\lambda_5}{\gamma} & 0 & -(s_1'+\lambda_4)\lambda_5 \\ \frac{1}{\mu\beta\gamma} & \frac{1}{\beta\gamma} & \frac{1}{\gamma} & 1 & s_1'+\lambda_4+\lambda_5 \end{pmatrix}.$$

(c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4, \lambda_5 \geqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix}$$

is a solution of this problem,where A is matrix (5.8) and a, b are zero vectors with dimension of 4×1.
(d) If $\lambda_2, \lambda_3 < 0$ and $\lambda_4, \lambda_5 \geqslant 0$, then by (6.4) we have $s_1' \geqslant 0$ then the nonnegative matrix

$$C = \begin{pmatrix} s_1' & -(\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}$$

is a solution of problem.
(e) If $\lambda_2, \lambda_3, \lambda_4 < 0$ and $\lambda_5 \geqslant 0$,then by (6.5) we have $s_1' + \lambda_4 \geqslant 0$, then the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix}$$

is a solution of this problem,where A is matrix (5.7) and a, b are zero vectors with dimension of 4×1.
Now let λ_2, λ_3 are complex conjugate pair and λ_4, λ_5 are real numbers. By conditions of theorem we consider the following cases.
(f) If $\lambda_4, \lambda_5 \geqslant 0$ and $\alpha_1 \leqslant 0$, then by (6.6) we have $s_1' \geqslant 0$, so the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix}$$

is a solution of our problem, where A is the matrix (5.9) and a, b are zero vector of dimension 4×1.
(g) If $\lambda_4, \lambda_5 \geqslant 0$, then by relation (6.7) we have $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$. Therefore the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix},$$

is solution, where A is the matrix (5.11) and a, b are zero vectors with dimension 4×1.|

(h) If $\lambda_4 < 0, \lambda_5 \geqslant 0$ and $\alpha_1 \leqslant 0$, then by (6.8) we must have $\acute{s_1} + \lambda_4 \geqslant 0$. Then the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix},$$

is solution, where A is the matrix (5.10) and a, b are zero vectors with dimension 4×1.
(i) If $\lambda_4 < 0, \lambda_5 \geqslant 0$ and $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$, then by (6.9) we must have $\alpha_3 + \lambda_4 \geqslant 0$, then by Theorem 2.1 we construct solution of problem. The matrix

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & \alpha_4 & 1 \\ 1 & \alpha_2\,(|\lambda_2|)^2 & \alpha_3 \end{pmatrix}$$

has eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ and the matrix

$$B = \begin{pmatrix} 0 & -\alpha_3\lambda_4 & 0 \\ 1 & \alpha_3 + \lambda_4 & 0 \\ 0 & 0 & \lambda_5 \end{pmatrix}$$

has the eigenvalues $\{\alpha_3, \lambda_4, \lambda_5\}$, which α_3 is Perron eigenvalue of B. Set $\beta = \sqrt{1 + (\lambda_4)^{-2}}$ then by combining two matrices A and B, the following matrix C is solution of our problem:

$$C = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & \alpha_4 & \frac{1}{\beta} & -\frac{1}{\lambda_4\beta} & 0 \\ \frac{1}{\beta} & \frac{\alpha_2(|\lambda_2|)^2}{\beta} & 0 & -\alpha_3\lambda_4 & 0 \\ -\frac{1}{\lambda_4\beta} & -\frac{\alpha_2(|\lambda_2|)^2}{\lambda_4\beta} & 1 & \alpha_3 + \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

(j) If $\lambda_4 < 0, \lambda_5 \geqslant 0$ and $\alpha_1 > 0$, then by (6.10) we have $(\alpha_2, \alpha_4 \geqslant 0, \alpha_3 \geqslant |\lambda_4|)$ or $(\alpha_2, \alpha_3 \geqslant 0, \alpha_4 \geqslant |\lambda_4|)$.

In first case, the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix}$$

is a solution of this problem,where A is matrix (5.12) and a, b are zero vectors with dimension of 4×1.
In second case, the nonnegative matrix

$$C = \begin{pmatrix} A & a \\ b^T & \lambda_5 \end{pmatrix}$$

is a solution of this problem,where A is matrix (5.13) and a, b are zero vectors with dimension of 4×1.

(k) If $\lambda_4, \lambda_5 < 0$ and $\alpha_1 < 0$, by (6.11), we have $s_1' + \lambda_4 + \lambda_5 \geqslant 0$, so $s_1' + \lambda_4 \geqslant 0$, then by Theorem 2.1, we construct a solution of problem. The nonnegative matrix (5.10) with the spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and the nonnegative matrix

$$B = \begin{pmatrix} 0 & -(s_1' + \lambda_4)\lambda_5 \\ 1 & s_1' + \lambda_4 + \lambda_5 \end{pmatrix}$$

with the spectrum $\sigma_2 = \{s_1' + \lambda_4, \lambda_5\}$ satisfy the conditions on matrices A and B of Theorem 2.1, respectively. The normalized vector corresponding to the Perron eigenvalue of $\lambda = s_1' + \lambda_4$ of nonnegative matrix B is $s = \left(\frac{-\lambda_5}{\sqrt{1+\lambda_5^2}}, \frac{1}{\sqrt{1+\lambda_5^2}}\right)^T$, setting $\beta = \sqrt{1 + \lambda_4^2}$ and $\gamma = \sqrt{1 + \lambda_5^2}$, then the following nonnegative matrix is solution

$$C = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & \frac{-\lambda_4}{\beta} & \frac{-\lambda_5}{\beta\gamma} & \frac{1}{\beta\gamma} \\ \frac{-\lambda_4}{\beta} & \frac{\alpha_1\lambda_4}{\beta} & 0 & \frac{s_1'\lambda_4\lambda_5}{\gamma} & \frac{-s_1'\lambda_4}{\gamma} \\ \frac{-\lambda_5}{\beta\gamma} & \frac{\alpha_1\lambda_5}{\beta\gamma} & \frac{-\lambda_5}{\gamma} & 0 & -(s_1' + \lambda_4)\lambda_5 \\ \frac{1}{\beta\gamma} & \frac{-\alpha_1}{\beta\gamma} & \frac{1}{\gamma} & 1 & s_1' + \lambda_4 + \lambda_5 \end{pmatrix}.$$

(l) If $\lambda_4, \lambda_5 < 0$, then by relation (6.12) we have
$(\alpha_2, \alpha_4 \geqslant 0, \alpha_3 + \lambda_4 \geqslant |\lambda_5|)$ or $(\alpha_2, \alpha_3 \geqslant 0, \alpha_4 + \lambda_4 \geqslant |\lambda_5|)$.

If $\alpha_2, \alpha_4 \geqslant 0, \alpha_3 + \lambda_4 \geqslant |\lambda_5|$, then it is obvious that the nonnegative matrix (5.12) with spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and the nonnegative matrix

$$B = \begin{pmatrix} 0 & -(\alpha_3 + \lambda_4)\lambda_5 \\ 1 & \alpha_3 + \lambda_4 + \lambda_5 \end{pmatrix},$$

with eigenvalues $\{\alpha_3, \lambda_4, \lambda_5\}$ satisfy the conditions of the matrices A, B in Theorem 2.1, respectively. In this case the Perron eigenvalue of $\lambda = \alpha_3 + \lambda_4$ has normalized eigenvector $s = \left(\frac{-\lambda_5}{\sqrt{1+\lambda_5^2}}, \frac{1}{\sqrt{1+\lambda_5^2}}\right)^T$. If we set $\beta = \sqrt{1 + \lambda_4^2}$ and $\gamma = \sqrt{1 + \lambda_5^2}$ then the nonnegative matrix

$$C = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & \alpha_4 & \frac{-\lambda_4}{\beta} & \frac{-\lambda_5}{\beta\gamma} & \frac{1}{\beta\gamma} \\ \frac{-\lambda_4}{\beta} & \frac{-\alpha_2\lambda_4|\lambda_2|^2}{\beta} & 0 & \frac{\alpha_3\lambda_4\lambda_5}{\gamma} & \frac{-\alpha_3\lambda_4}{\gamma} \\ \frac{-\lambda_5}{\beta\gamma} & \frac{-\lambda_5\alpha_2|\lambda_2|^2}{\beta\gamma} & \frac{-\lambda_5}{\gamma} & 0 & -(\alpha_3 + \lambda_4)\lambda_5 \\ \frac{1}{\beta\gamma} & \frac{\alpha_2|\lambda_2|^2}{\beta\gamma} & \frac{1}{\gamma} & 1 & \alpha_3 + \lambda_4 + \lambda_5 \end{pmatrix}$$

is a solution of our problem.
If $\alpha_2, \alpha_3 \geqslant 0, \alpha_4 + \lambda_4 \geqslant |\lambda_5|$, then by similar process the matrix (5.13) and matrix

$$B = \begin{pmatrix} 0 & -(\alpha_4 + \lambda_4)\lambda_5 \\ 1 & \alpha_4 + \lambda_4 + \lambda_5 \end{pmatrix},$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\sigma_2 = \{\alpha_4 + \lambda_4, \lambda_5\}$ satisfy conditions of A, B in Theorem 2.1, respectively, and consequently the nonnegative matrix

$$C = \begin{pmatrix} \alpha_3 & 1 & \frac{-\lambda_4\alpha_2|\lambda_2|^2}{\beta} & \frac{-\lambda_5\alpha_2|\lambda_2|^2}{\beta\gamma} & \frac{\alpha_2|\lambda_2|^2}{\beta\gamma} \\ 0 & 0 & \frac{-p\lambda_4}{\beta} & \frac{-p\lambda_5}{\beta\gamma} & \frac{p}{\beta\gamma} \\ \frac{-\lambda_4}{\beta} & 0 & 0 & \frac{\alpha_4\lambda_4\lambda_5}{\beta} & \frac{-\alpha_4\lambda_4}{\gamma} \\ \frac{-\lambda_5}{\beta\gamma} & 0 & \frac{-\lambda_5}{\beta\gamma} & 0 & -(\alpha_4+\lambda_4)\lambda_5 \\ \frac{1}{\beta\gamma} & 0 & \frac{1}{\beta} & 1 & \alpha_4+\lambda_4+\lambda_5 \end{pmatrix}$$

is a solution of our problem.

Let λ_2, λ_3 and λ_4, λ_5 are two complex conjugate pairs, then by conditions of theorem we consider the following cases.

(m) If $\alpha_1, \alpha_5 \leqslant 0$, then by (6.13) we have $s_1' \geqslant 0$. For finding solution, first, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_1 & s_1' \end{pmatrix}, \quad B = \begin{pmatrix} 0 & s_1'|\lambda_4|^2 & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_5 & s_1'+\lambda_4+\lambda_5 \end{pmatrix}$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{s_1', \lambda_4, \lambda_5\}$, respectively. Whereas $s_1' \geqslant 0$ and $\alpha_5 \leqslant 0$, we have $\lambda_4 = -x + iy$ and $\lambda_5 = -x - iy$, where $x \geqslant 0$. Since $s_1' + \lambda_4 + \lambda_5 \geqslant 0$, therefore

$$0 \leqslant s_1' + \lambda_4 + \lambda_5 \rightarrow s_1' \geqslant 2x.$$

By above relation and $\alpha_5 \leqslant 0$, we have

$$\alpha_5 = s_1'(-2x) + x^2 + y^2 \leqslant 0 \rightarrow x^2 + y^2 \leqslant s_1'(2x) \leqslant {s_1'}^2 \rightarrow |\lambda_4| \leqslant s_1'.$$

Thus s_1' is the Perron eigenvalue of nonnegative matrix B. If $\beta = \sqrt{1 + {s_1'}^2 + |\lambda_4|^4}$, then the normalized eigenvector of s_1' is $s = \left(\frac{|\lambda_4|^2}{\beta}, \frac{1}{\beta}, \frac{s_1'}{\beta}\right)^T$. It is clear that the matrices A and B satisfy Theorem 2.1, then the following nonnegative matrix is solution,

$$C = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & \frac{|\lambda_4|^2}{\beta} & \frac{1}{\beta} & \frac{s_1'}{\beta} \\ \frac{|\lambda_4|^2}{\beta} & \frac{-\alpha_1|\lambda_4|^2}{\beta} & 0 & s_1'|\lambda_4|^2 & 0 \\ \frac{1}{\beta} & \frac{-\alpha_1}{\beta} & 0 & 0 & 1 \\ \frac{s_1'}{\beta} & \frac{-s_1'\alpha_1}{\beta} & 0 & -\alpha_5 & s_1'+\lambda_4+\lambda_5 \end{pmatrix}.$$

(n) If $\alpha_1 \leqslant 0$ and $\alpha_5 > 0$, then by (6.14) we have $\alpha_6, \alpha_7, \alpha_{14} \geqslant 0$.

First, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_1 & s_1' \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_7 & \alpha_6|\lambda_4|^2 & 1 \\ 1 & \alpha_{14} & 0 \\ 0 & s_1'|\lambda_4|^2 & 0 \end{pmatrix}.$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{s_1', \lambda_4, \lambda_5\}$, respectively. It is trivial that the matrices above satisfy Theorem 2.1. Now by setting $\beta = \sqrt{1 + (1+\alpha_6)^2 + |\lambda_4|^4}$, we have the normalized eigenvector associated to Perron eigenvalue s_1' has form $s = \left(\frac{\alpha_6+1}{\beta}, \frac{1}{\beta}, \frac{|\lambda_4|^2}{\beta}\right)^T$. Therefore by Theorem 2.1 the nonnegative matrix

$$C=\begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\alpha_6}{\beta} & \frac{1}{\beta} & \frac{|\lambda_4|^2}{\beta} \\ \frac{1+\alpha_6}{\beta} & \frac{-(1+\alpha_6)\alpha_1}{\beta} & \alpha_7 & \alpha_6|\lambda_4|^2 & 1 \\ \frac{1}{\beta} & \frac{-\alpha_1}{\beta} & 1 & \alpha_{14} & 0 \\ \frac{|\lambda_4|^2}{\beta} & \frac{-\alpha_1|\lambda_4|^2}{\beta} & 0 & \acute{s_1}|\lambda_4|^2 & 0 \end{pmatrix}$$

is a solution of problem.
(o) If $\alpha_1 > 0$ and $\alpha_8 \leqslant 0$, then by (6.15) we have $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$ and $\alpha_3 + \lambda_4 + \lambda_5 \geqslant 0$.
For construction solution of problem in this here, First, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A=\begin{pmatrix} 0 & p & 0 \\ 0 & \alpha_4 & 1 \\ 1 & \alpha_2|\lambda_2|^2 & \alpha_3 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & \alpha_3|\lambda_4|^2 & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_8 & \alpha_3+\lambda_4+\lambda_5 \end{pmatrix},$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_3, \lambda_4, \lambda_5\}$, respectively. Since $\alpha_3 \geqslant 0$ and $\alpha_8 \leqslant 0$, we have $\lambda_4 = -x + iy$ and $\lambda_5 = -x - iy$, where $x \geqslant 0$. Whereas $\alpha_3 + \lambda_4 + \lambda_5 \geqslant 0$, therefore

$$0 \leqslant \alpha_3 + \lambda_4 + \lambda_5 = \alpha_3 - 2x \to \alpha_3 \geqslant 2x,$$

By above relation and $\alpha_8 \leqslant 0$, we have

$$\alpha_8 = \alpha_3(-2x) + x^2 + y^2 \leqslant 0 \to |\lambda_4| \leqslant \alpha_3.$$

Then α_3 is the Perron eigenvalue of nonnegative matrix B. By setting $\beta = \sqrt{1+\alpha_3^2+|\lambda_4|^4}$, the normalized eigenvector associated to α_3 is $s = \left(\frac{|\lambda_4|^2}{\beta}, \frac{1}{\beta}, \frac{\alpha_3}{\beta}\right)^T$. It is clear that the matrices A and B satisfy the condition of Theorem 2.1, then the following nonnegative matrix is solution,

$$C=\begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & \alpha_4 & \frac{|\lambda_4|^2}{\beta} & \frac{1}{\beta} & \frac{\alpha_3}{\beta} \\ \frac{|\lambda_4|^2}{\beta} & \frac{|\lambda_2|^2\alpha_2|\lambda_4|^2}{\beta} & 0 & \alpha_3|\lambda_4|^2 & 0 \\ \frac{1}{\beta} & \frac{\alpha_2|\lambda_2|^2}{\beta} & 0 & 0 & 1 \\ \frac{\alpha_3}{\beta} & \frac{\alpha_2|\lambda_2|^2\alpha_3}{\beta} & 1 & -\alpha_8 & \alpha_3+\lambda_4+\lambda_5 \end{pmatrix}.$$

(p) If $\alpha_1, \alpha_8 > 0$, then by (6.16) we have $\alpha_2, \alpha_4, \alpha_9, \alpha_{10}, \alpha_{14} \geqslant 0$.
In here, first, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A=\begin{pmatrix} 0 & p & 0 \\ 0 & \alpha_4 & 1 \\ 1 & \alpha_2|\lambda_2|^2 & \alpha_3 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & \alpha_3|\lambda_4|^2 & 0 \\ 0 & \alpha_{14} & 1 \\ 1 & \alpha_9|\lambda_4|^2 & \alpha_{10} \end{pmatrix}.$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_3, \lambda_4, \lambda_5\}$, respectively. Now we set $\beta = \sqrt{1+(1+\alpha_9)^2+|\lambda_4|^4}$, then the normalized eigenvector associated to Perron eigenvalue α_3 is in the form $s = \left(\frac{|\lambda_4|^2}{\beta}, \frac{1}{\beta}, \frac{\alpha_9+1}{\beta}\right)^T$. It is clear that the matrices A, B satisfy the condition of Theorem 2.1. Therefore by this theorem the nonnegative matrix

$$C=\begin{pmatrix} 0 & p & 0 & 0 & 0\\ 0 & \alpha_4 & \frac{|\lambda_4|^2}{\beta} & \frac{1}{\beta} & \frac{\alpha_9+1}{\beta}\\ \frac{|\lambda_4|^2}{\beta} & \frac{|\lambda_4|^2\alpha_2|\lambda_2|^2}{\beta} & 0 & \alpha_3|\lambda_4|^2 & 0\\ \frac{1}{\beta} & \frac{\alpha_2|\lambda_2|^2}{\beta} & 0 & \alpha_{14} & 1\\ \frac{\alpha_9+1}{\beta} & \frac{(\alpha_9+1)\alpha_2|\lambda_2|^2}{\beta} & 1 & \alpha_9|\lambda_4|^2 & \alpha_{10} \end{pmatrix}$$

is a solution of problem.
(q) If $\alpha_1 > 0$ and $\alpha_{11} \leqslant 0$, then by (6.17) we have $\alpha_2, \alpha_3, \alpha_4 \geqslant 0$ and $\alpha_4 + \lambda_4 + \lambda_5 \geqslant 0$.
For construction solution of problem , first, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A=\begin{pmatrix} \alpha_3 & 1 & \alpha_2|\lambda_2|^2\\ 0 & 0 & p\\ 1 & 0 & \alpha_4 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & \alpha_4|\lambda_4|^2 & 0\\ 0 & 0 & 1\\ 1 & -\alpha_{11} & \alpha_4+\lambda_4+\lambda_5 \end{pmatrix}$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_4, \lambda_4, \lambda_5\}$, respectively. Since $\alpha_4 \geqslant 0$ and $\alpha_{11} \leqslant 0$, then $\lambda_4 = -x + iy$ and $\lambda_5 = -x - iy$, where $x \geqslant 0$. Whereas $\alpha_4 + \lambda_4 + \lambda_5 \geqslant 0$, therefore

$$0 \leqslant \alpha_4 + \lambda_4 + \lambda_5 = \alpha_4 - 2x \to \alpha_4 \geqslant 2x.$$

By above relation and $\alpha_{11} \leqslant 0$ we have

$$\alpha_{11} = \alpha_4(-2x) + x^2 + y^2 \leqslant 0 \to |\lambda_4| \leqslant \alpha_4.$$

Then α_4 is the Perron eigenvalue of nonnegative matrix B. By setting $\beta = \sqrt{1+\alpha_4^2+|\lambda_4|^4}$, the normalized eigenvector associated to α_4 is $s = \left(\frac{|\lambda_4|^2}{\beta}, \frac{1}{\beta}, \frac{\alpha_4}{\beta}\right)^T$. By noting that the matrices A and B satisfy the condition of Theorem 2.1, then by this theorem the following nonnegative matrix is solution,

$$C=\begin{pmatrix} \alpha_3 & 1 & \frac{\alpha_2|\lambda_2|^2|\lambda_4|^2}{\beta} & \frac{\alpha_2|\lambda_2|^2}{\beta} & \frac{\alpha_2|\lambda_2|^2\alpha_4}{\beta}\\ 0 & 0 & \frac{p|\lambda_4|^2}{\beta} & \frac{p}{\beta} & \frac{p\alpha_4}{\beta}\\ \frac{|\lambda_4|^2}{\beta} & 0 & 0 & \alpha_4|\lambda_4|^2 & 0\\ \frac{1}{\beta} & 0 & 0 & 0 & 1\\ \frac{\alpha_4}{\beta} & 0 & 1 & -\alpha_{11} & \alpha_4+\lambda_4+\lambda_5 \end{pmatrix}.$$

(r) If $\alpha_1, \alpha_{11} > 0$, then by (6.18) we have $\alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{14} \geqslant 0$.

In here, first, consider the following nonnegative matrices which arose in the discussion of the case $n = 3$,

$$A=\begin{pmatrix} \alpha_3 & 1 & \alpha_2|\lambda_2|^2\\ 0 & 0 & p\\ 1 & 0 & \alpha_4 \end{pmatrix}, \quad B=\begin{pmatrix} \alpha_{13} & \alpha_{12}|\lambda_4|^2 & 1\\ 1 & \alpha_{14} & 0\\ 0 & \alpha_4|\lambda_4|^2 & 0 \end{pmatrix}.$$

with spectrum $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\sigma_2 = \{\alpha_4, \lambda_4, \lambda_5\}$, respectively. Now we set $\beta = \sqrt{1+(1+\alpha_{12})^2+|\lambda_4|^4}$, then the normalized eigenvector associated to Perron eigenvalue α_4 is in the form $s = \left(\frac{\alpha_{12}+1}{\beta}, \frac{1}{\beta}, \frac{|\lambda_4|^2}{\beta}\right)^T$. It is clear that the matrices A, B satisfy the condition of Theorem

2.1. Therefore by this theorem the nonnegative matrix

$$C = \begin{pmatrix} \alpha_3 & 1 & \frac{\alpha_2|\lambda_2|^2(\alpha_{12}+1)}{\beta} & \frac{\alpha_2|\lambda_2|^2}{\beta} & \frac{\alpha_2|\lambda_2|^2|\lambda_4|^2}{\beta} \\ 0 & 0 & \frac{p(\alpha_{12}+1)}{\beta} & \frac{p}{\beta} & \frac{p|\lambda_4|^2}{\beta} \\ \frac{\alpha_{12}+1}{\beta} & 0 & \alpha_{13} & \alpha_{12}|\lambda_4|^2 & 1 \\ \frac{1}{\beta} & 0 & 1 & \alpha_{14} & 0 \\ \frac{|\lambda_4|^2}{\beta} & 0 & 0 & \alpha_4|\lambda_4|^2 & 0 \end{pmatrix}$$

is a solution of problem. □

Remark 6.2. Assume given

$$\sigma = \{\lambda_1 = 25, \lambda_2 = 1 + 3i, \lambda_2 = 1 - 3i, \lambda_4 = 0.5 - i, \lambda_5 = 0.5 + i\},$$

then we have

$$F(\lambda) = \lambda^5 - 28.0\,\lambda^4 + 88.25\,\lambda^3 - 343.75\,\lambda^2 + 325.0\,\lambda - 312.50,$$

since some coefficients of λ is positive, thus we can not find the nonnegative companion matrix with spectrum σ. On the other hand all condition of case (r) in above theorem are satisfied, then we find the following nonnegative matrix that σ is its spectrum

$$C = \begin{pmatrix} 15 & 1 & 118.4146234 & 12.14508958 & 15.181362 \\ 0 & 0 & 246.6971322 & 25.30226997 & 31.62783748 \\ 0.9867885287 & 0 & 10.75 & 10.9375 & 1 \\ 0.10120908 & 0 & 1 & 2.25 & 0 \\ 0.12651135 & 0 & 0 & 15 & 0 \end{pmatrix},$$

and if we assume that

$$\sigma = \{\lambda_1 = 25, \lambda_2 = 1 + 3i, \lambda_2 = 1 - 3i, \lambda_4 = 6, \lambda_5 = 7\},$$

then we have

$$F(\lambda) = \lambda^5 - 40\,\lambda^4 + 453\,\lambda^3 - 2164\,\lambda^2 + 5770\,\lambda - 10500,$$

then we do not have a companion matrix for this σ, but by case (g) of above theorem we have the following nonnegative solution

$$C = \begin{pmatrix} 15 & 120 & 1 & 0 & 0 \\ 1 & 12 & 0 & 0 & 0 \\ 0 & 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

If given

$$\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = \{20, -3 - i, -3 + i, -6, -7\},$$

then we have

$$\begin{aligned} F(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)(\lambda-\lambda_5) \\ &= \lambda^5-\lambda^4-250\lambda^3-2218\lambda^2-7220\lambda-8400, \end{aligned}$$

although the nonnegative companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 8400 & 7220 & 2218 & 250 & 1 \end{pmatrix}$$

has spectrum σ, we find by Theorem 2.1 and Theorem 6.1 (case (l)) another nonnegative matrix with spectrum σ. The 4×4

$$\begin{aligned} A &= \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & -\frac{\lambda_4}{\sqrt{1+\lambda_4{}^2}} & \frac{1}{\sqrt{1+\lambda_4{}^2}} \\ -\frac{\lambda_4}{\sqrt{1+\lambda_4{}^2}} & \frac{\alpha_1\lambda_4}{\sqrt{1+\lambda_4{}^2}} & 0 & -(\lambda_1+\lambda_2+\lambda_3)\,\lambda_4 \\ \frac{1}{\sqrt{1+\lambda_4{}^2}} & -\frac{\alpha_1}{\sqrt{1+\lambda_4{}^2}} & 1 & \lambda_1+\lambda_2+\lambda_3+\lambda_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 200 & 0 & 0 \\ 0 & 0 & \frac{6}{37}\sqrt{37} & 1/37\sqrt{37} \\ \frac{6}{37}\sqrt{37} & \frac{660}{37}\sqrt{37} & 0 & 84 \\ 1/37\sqrt{37} & \frac{110}{37}\sqrt{37} & 1 & 8 \end{pmatrix} \end{aligned}$$

has spectrum $\{20, -3+i, -3-i, -6\}$ and the matrix

$$B = \begin{pmatrix} 0 & -(\acute{s_1}+\lambda_4)\,\lambda_5 \\ 1 & \acute{s_1}+\lambda_4+\lambda_5 \end{pmatrix} = \begin{pmatrix} 0 & 56 \\ 1 & 1 \end{pmatrix}$$

has eigenvalue 8 and -7. By Theorem 2.1 with combining two matrices A and B, we have the following matrix C, that has spectrum σ

$$C = \begin{pmatrix} 0 & 200 & 0 & 0 & 0 \\ 0 & 0 & \frac{6}{37}\sqrt{37} & \frac{7}{370}\sqrt{37}\sqrt{2} & \frac{1}{370}\sqrt{37}\sqrt{2} \\ \frac{6}{37}\sqrt{37} & \frac{660}{37}\sqrt{37} & 0 & \frac{294}{5}\sqrt{2} & \frac{42}{5}\sqrt{2} \\ \frac{7}{370}\sqrt{37}\sqrt{2} & \frac{77}{37}\sqrt{37}\sqrt{2} & \frac{7}{10}\sqrt{2} & 0 & 56 \\ \frac{1}{370}\sqrt{37}\sqrt{2} & \frac{11}{37}\sqrt{37}\sqrt{2} & 1/10\sqrt{2} & 1 & 1 \end{pmatrix}.$$

Remark 6.3. Assume given

$$\sigma = \{\lambda_1 = 11, \lambda_2 = -3 + i, \lambda_2 = -3 - i, \lambda_4 = 6, \lambda_5 = 7\},$$

then we have

$$F(\lambda) = \lambda^5 - 18\,\lambda^4 + 51\,\lambda^3 + 408\,\lambda^2 - 922\,\lambda - 4620.$$

The eigenvalues of a real 5×5 circulant have the form $a, b + ic, b - ic, p + iq, p - iq$, for some real numbers a, b, c, p, q, so it is obvious from $6 \neq 7$ that σ is not the spectrum of a nonnegative circulant. It is worth noting that $\tau = (6, -3 + i, -3 - i)$ is realizable by a nonnegative companion matrix C (and also by a nonnegative circulant) and therefore the spectrum $(m + 6, -3 + i, -3 - i, 6, 7)$ is realizable by the direct sum of $mI_3 + C$ and the diagonal matrix $\mathrm{diag}(6, 7)$ for all $m \geq 0$. On the other hand by case(f) and case(g) of Theorem 6.1 the following nonnegative matrices has spectrum σ:

$$C_1 = \begin{pmatrix} \alpha_3 & \alpha_2\,(|\lambda_2|)^2 & 1 & 0 & 0 \\ 1 & \alpha_4 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix} = \begin{pmatrix} 1 & 60 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 110 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -\alpha_1 & s_1' & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix} = \begin{pmatrix} 0 & 110 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 56 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Now assume given

$$\sigma = \{\lambda_1 = 12, \lambda_2 = -3 + i, \lambda_2 = -3 - i, \lambda_4 = -6, \lambda_5 = 7\},$$

then we have

$$F(\lambda) = \lambda^5 - 7\,\lambda^4 - 98\,\lambda^3 + 194\,\lambda^2 + 2724\,\lambda + 5040.$$

By the same reason there is no nonnegative circulant matrix that realize σ, but by case(h) of Theorem 6.1 the following nonnegative matrix is realized σ:

$$C = \begin{pmatrix} 0 & 120 & 0 & 0 & 0 \\ 0 & 0 & \frac{6}{37}\sqrt{37} & 1/37\sqrt{37} & 0 \\ \frac{6}{37}\sqrt{37} & \frac{372}{37}\sqrt{37} & 0 & 36 & 0 \\ 1/37\sqrt{37} & \frac{62}{37}\sqrt{37} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Remark 6.4. It is worth noting that in cases (a)-(h) of Theorem 6.1, if σ is realizable by a nonnegative matrix, then the all main conditions of $\alpha\sigma$ for $\alpha \geq 0$ are not change, therefore the set $\alpha\sigma$ also is realizable by a nonnegative matrix in a same method. Furthermore if $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ is realizable by a nonnegative matrix, then with consideration the main conditions of $\sigma' = \{\lambda_1 + m, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, for $m \geq 0$, we see that all conditions of Theorem 6.1 in all cases are satisfied, then there is a nonnegative matrix that realize σ' in same method that realized σ.

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